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Interval max-plus systems of linear equations

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ABSTRACT

In this paper, we shall deal with solvability of interval systems of linear equations in max-plus algebra. Max-plus algebra is an algebraic structure in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$.

The notation $\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}$ represents an interval system of linear equations, where $\boldsymbol{A} = \langle \underline{A}, \overline{A} \rangle$, $\boldsymbol{b} = \langle \underline{b}, \overline{b} \rangle$ are given interval matrix and interval vector, respectively, and a solution is from a given interval vector $\boldsymbol{x} = \langle \underline{x}, \overline{x} \rangle$. We can define several types of solvability of interval systems. In this paper, we define six types of solvability of interval max-plus systems of linear equations and give necessary and sufficient conditions for them.

1. Introduction

Problems on algebraic structures, in which the pairs of operations (max, +) or (max, min) replace addition and multiplication of the classical linear algebra, appear in the literature approximately since the sixties of the last century (see e.g. [2,12]). A systematic theory of such algebraic structures was published probably for the first time in [2]. One of the problems we can deal with is solving of systems of linear equations, which are useful for modeling of discrete dynamic systems, scheduling or graph theory. Among interesting real-life applications let us mention, e.g., a large scale model of Dutch railway network or synchronizing traffic lights in Delft [9].

In practice it may often happen that a given system of linear equations is not solvable. One of the methods of restoring solvability is to replace elements of matrix A and vector b by intervals of possible

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values. The resulting systems are the so-called interval systems of linear equations. We can define several solvability concepts for interval systems of linear equations. Rohn [11] dealt with solvability of interval systems of linear equations over the classical algebra. An interesting approach to interval computations in max–min algebra was published in [3,10]. Interval systems of max-plus linear equations have been studied by Cechlárová and Cuninghame-Green in [1]. Other solvability concepts are studied in [4,5,7]. In this paper, we shall deal with interval systems of linear equations with bounded solution. We define several solvability concepts and give necessary and sufficient conditions for them.

2. Background of the problem

Let $(B, \oplus, \otimes)$ be an algebraic structure with two binary operations. $(B, \oplus, \otimes)$ is called the *max-plus algebra*, if

$$B = \mathbb{R} \cup \{\varepsilon\}, \quad a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b,$$

where $\varepsilon = -\infty$.

Let m, n be given positive integers. Denote by M and N the sets of indices $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, respectively. The set of all $m \times n$ *matrices* over B is denoted by $B(m, n)$ and the set of all column *n-vectors* over B by $B(n)$.

We shall consider the *ordering* $\leqslant$ on the sets $B(m, n)$ and $B(n)$ defined as follows:

- for $A, C \in B(m, n) : \ A \leqslant C$ if $a_{ij} \leqslant c_{ij}$ for all $i \in M, \ j \in N$,
- for $x, y \in B(n) : \ x \leqslant y$ if $x_j \leqslant y_j$ for all $j \in N$.

The matrix operations in max-plus algebra are defined formally in the same manner (with respect to $\oplus, \otimes$) as matrix operations in the classical linear algebra.

It is easy to see that for each $A, C \in B(m, n)$ and for each $x, y \in B(n)$ the implication

$$\text{if } A \leqslant C \text{ and } x \leqslant y, \text{ then } A \otimes x \leqslant C \otimes y$$

holds. We shall call this property the *monotonicity* of the operation $\otimes$.

For a given matrix interval $\boldsymbol{A} = \langle \underline{A}, \overline{A} \rangle$ with $\underline{A}, \overline{A} \in B(m, n), \ \underline{A} \leqslant \overline{A}$ and a given vector interval $\boldsymbol{b} = \langle \underline{b}, \overline{b} \rangle$ with $\underline{b}, \overline{b} \in B(m), \ \underline{b} \leqslant \overline{b}$ the notation

$$\boldsymbol{A} \otimes x = \boldsymbol{b} \tag{1}$$

represents the set of all systems of linear max-plus equations of the form

$$A \otimes x = b, \tag{2}$$

where $A \in \boldsymbol{A}, \ b \in \boldsymbol{b}$.

The set $\boldsymbol{A} \otimes x = \boldsymbol{b}$ will be called an *interval system* of max-plus linear equations. Each system of the form (2) is said to be a *subsystem* of system (1), if $A \in \boldsymbol{A}, \ b \in \boldsymbol{b}$. We shall suppose that $\underline{b}_i \neq \varepsilon$ for each $i \in M$ and for each $i \in M$ there exists $j \in N$ such that $\underline{a}_{ij} \neq \varepsilon$.

We shall consider the solvability of interval system (1) on the ground of the solvability of its subsystems. If we ask for the solvability of at least one subsystem we speak about the *weak* solvability of interval system (1). On the other hand the *strong* solvability requires solvability of all subsystems. Another possibility is the *tolerance* solvability which asks for the existence of a vector $x \in B(n)$ such that for each $A \in \boldsymbol{A}$ the product $A \otimes x$ belongs to $\boldsymbol{b}$. In this way we can define various solvability concepts. Table 1 contains the list of all up to now studied types of solvability of (1) in max-plus algebra. Some of them have been studied in the classical algebra in [11]. The paper [7], which has been not published yet, deals with the T4 solvability of (1). An iterative algorithm for testing the T4 solvability is presented. We will not use these results in the present paper.

The solvability concepts which lead to trivial conditions and the solvability concepts where the quantifier $\forall$ stands by x are omitted there. The reason is that if the entries of x are upper unbounded

Table 1
Solvability concepts of (1).

Solvabilityconcept	*Definition*
Weak solvability [1]	$(\exists x\in B(n))(\exists A\in \boldsymbol{A})(\exists b\in \boldsymbol{b}):A\otimes x=b$
Strong solvability [1]	$(\forall A\in \boldsymbol{A})(\forall b\in \boldsymbol{b})(\exists x\in B(n)):A\otimes x=b$
Tolerance solvability [1]	$(\exists x\in B(n))(\forall A\in \boldsymbol{A})(\exists b\in \boldsymbol{b}):A\otimes x=b$
Weak tolerance solvability [4]	$(\forall A\in \boldsymbol{A})(\exists x\in B(n))(\exists b\in \boldsymbol{b}):A\otimes x=b$
Control solvability [5]	$(\exists x\in B(n))(\forall b\in \boldsymbol{b})(\exists A\in \boldsymbol{A}):A\otimes x=b$
Weak control solvability [5]	$(\forall b\in \boldsymbol{b})(\exists x\in B(n))(\exists A\in \boldsymbol{A}):A\otimes x=b$
Universal solvability [4]	$(\exists x\in B(n))(\forall b\in \boldsymbol{b})(\forall A\in \boldsymbol{A}):A\otimes x=b$
Weak universal solvability [5]	$(\forall b\in \boldsymbol{b})(\exists x\in B(n))(\forall A\in \boldsymbol{A}):A\otimes x=b$
T4 solvability [7]	$(\exists b\in \boldsymbol{b})(\exists x\in B(n))(\forall A\in \boldsymbol{A}):A\otimes x=b$

Table 2
Solvability concepts of (3).

Solvabilityconcept	*Definition*
T1 solvability	$(\exists A\in \boldsymbol{A})(\forall x\in \boldsymbol{x})(\exists b\in \boldsymbol{b}):A\otimes x=b$
T2 solvability	$(\forall x\in \boldsymbol{x})(\exists A\in \boldsymbol{A})(\exists b\in \boldsymbol{b}):A\otimes x=b$
T3 solvability [6]	$(\forall x\in \boldsymbol{x}(\exists b\in \boldsymbol{b}))(\forall A\in \boldsymbol{A}):A\otimes x=b$
T5 solvability [6]	$(\forall x\in \boldsymbol{x})(\forall A\in \boldsymbol{A})(\exists b\in \boldsymbol{b}):A\otimes x=b$
weak T6 solvability	$(\exists b\in \boldsymbol{b})(\forall x\in \boldsymbol{x})(\exists A\in \boldsymbol{A}):A\otimes x=b$
strong T6 solvability	$(\forall b\in \boldsymbol{b})(\forall x\in \boldsymbol{x})(\exists A\in \boldsymbol{A}):A\otimes x=b$
weak T7 solvability	$(\exists b\in \boldsymbol{b})(\exists A\in \boldsymbol{A})(\forall x\in \boldsymbol{x}):A\otimes x=b$
strong T7 solvability	$(\forall b\in \boldsymbol{b})(\exists A\in \boldsymbol{A})(\forall x\in \boldsymbol{x}):A\otimes x=b$
T8 solvability [6]	$(\forall A\in \boldsymbol{A})(\exists b\in \boldsymbol{b})(\forall x\in \boldsymbol{x}):A\otimes x=b$
T9 solvability [6]	$(\exists b\in \boldsymbol{b})(\forall A\in \boldsymbol{A})(\forall x\in \boldsymbol{x}):A\otimes x=b$

then for each $A \in \boldsymbol{A}$ the product $A \otimes x$ is upper unbounded, too. On the other hand, the entries of a vector $b \in \boldsymbol{b}$ are finite. Consequently, there is no interval system which satisfies the definition of some solvability concept with $\forall x \in B(n)$.

This leads to the research of interval systems such that solutions of subsystems of (1) cannot be arbitrary, but they are required to be from a given interval vector $\boldsymbol{x} = \langle \underline{x}, \overline{x}\rangle$, $\underline{x},\ \overline{x} \in B(n)$, $\underline{x} \leqslant \overline{x}$, where $\underline{x}_j \neq \varepsilon$ for each $j \in N$.

Denote by

$$\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b} \tag{3}$$

an *interval system with bounded solution* $x \in \boldsymbol{x}$. In fact, we can adapt solvability concepts of interval system (1) defined in Table 1 for interval system (3), too. The published results for interval system (1) change only slightly if we replace interval system (1) with (3).

Now, we shall define some more types of solvability of (3), which arise by involving x with quantifier $\forall$ (see Table 2). Similarly as above, Table 2 does not contain solvability concepts with trivial necessary and sufficient conditions.

3. Known results

Necessary and sufficient conditions for the solvability concepts of (3), which have been studied in [6], are presented in this section. These results will not be used in the present paper.

Theorem 1 [6]. *Interval system (3) is T5 solvable if and only if*

$$\underline{A} \otimes \underline{x} \geqslant \underline{b}, \tag{4}$$

$$\overline{A} \otimes \overline{x} \leqslant \overline{b}. \tag{5}$$

For given indices $i \in M,\ j \in N$ denote the vector $x^{(j)} = (x_k^{(j)})$ and the matrix $A^{(ij)} = (a_{kl}^{(ij)})$ as follows:

$$x_k^{(j)} = \begin{cases} \overline{x}_k & \text{for } k = j, \\ \underline{x}_k & \text{otherwise.} \end{cases} \qquad a_{kl}^{(ij)} = \begin{cases} \overline{a}_{kl} & \text{for } k = i,\ l = j, \\ \underline{a}_{kl} & \text{otherwise.} \end{cases}$$

The following lemma is of importance for the proofs of Theorem 2 and Theorem 3.

Lemma 1 [6]. *Let $x \in \boldsymbol{x},\ A \in \boldsymbol{A}$ be arbitrary. Then there exist $\alpha_i,\ \beta_{ij} \in B$ such that*

(i) $x = \bigoplus_{j \in N} \alpha_j \otimes x^{(j)}$,

(ii) $A = \bigoplus_{j \in N} \beta_{ij} \otimes A^{(ij)}$.

Proof. It is easy to see that $\alpha_j = x_j - \overline{x}_j$ and $\beta_{ij} = a_{ij} - \overline{a}_{ij}$ for each $i \in M,\ j \in N$. □

Theorem 2 [6]. *Interval system (3) is T3 solvable if and only if interval system (3) is T5 solvable and*

$$\underline{A} \otimes x^{(j)} = \overline{A} \otimes x^{(j)} \tag{6}$$

for each $j \in N$.

Theorem 3 [6]. *Interval system (3) is T8 solvable if and only if interval system (3) is T5 solvable and*

$$A^{(ij)} \otimes \underline{x} = A^{(ij)} \otimes \overline{x} \tag{7}$$

for each $i \in M,\ j \in N$.

Theorem 4 [6]. *Interval system (3) is T9 solvable if and only if interval system (3) is T5 solvable and*

$$\underline{A} \otimes \underline{x} = \overline{A} \otimes \overline{x}.$$

4. T1 and T2 solvability

Necessary and sufficient conditions for the T1 and T2 solvability are proved in this section. To give an equivalent condition for the T2 solvability we shall use the notion of a possible solution defined in [1].

Definition 1. A vector $x \in B(n)$ is a possible solution of interval system (1) if there exist $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $A \otimes x = b$.

The proposition of the following theorem has been known for many years in classical linear algebra due to [8].

Theorem 5 [1]. *A vector $x \in B(n)$ is a possible solution of interval system (1) if and only if*

$$\overline{A} \otimes x \geqslant \underline{b}, \tag{8}$$

$$\underline{A} \otimes x \leqslant \overline{b}. \tag{9}$$

It is clear that a possible solution can be defined in the same manner for interval system (3) and Theorem 5 is valid in this case, too.

Theorem 6. *Interval system (3) is T2 solvable if and only if*

$$\overline{A} \otimes \underline{x} \geqslant \underline{b}, \tag{10}$$

$$\underline{A} \otimes \overline{x} \leqslant \overline{b}. \tag{11}$$

Proof. According to Definition 1 interval system (3) is T2 solvable if and only if each vector $x \in \boldsymbol{x}$ is a possible solution of (3). Inequality (8) is fulfilled for each $x \in \boldsymbol{x}$ if and only if (8) holds true for $\underline{x}$, so we get inequality (10). Similarly we get (11). □

Lemma 2. *Interval system (3) is T1 solvable if and only if there exists a matrix $A \in \boldsymbol{A}$ such that*

$$A \otimes \underline{x} \geqslant \underline{b}, \tag{12}$$

$$A \otimes \overline{x} \leqslant \overline{b}. \tag{13}$$

Proof. T1 solvability means that there exists $A \in \boldsymbol{A}$ such that $A \otimes x \in \langle \underline{b}, \overline{b} \rangle$ for each $x \in \boldsymbol{x}$, i.e. $A \otimes x \geqslant \underline{b}$ and $A \otimes x \leqslant \overline{b}$ for each $x \in \boldsymbol{x}$. First (second) of these inequalities is satisfied for each $x \in \boldsymbol{x}$ if and only if it holds for $\underline{x}$ ($\overline{x}$), which is equivalent to the system of inequalities (12), (13). □

Lemma 2 does not provide an efficient algorithm for checking the T1 solvability. For this reason, we define a matrix $A^* = (a^*_{ij})$ as follows:

$$a^*_{ij} = \min\{\overline{b}_i - \overline{x}_j, \overline{a}_{ij}\} \tag{14}$$

for each $i \in M,\ j \in N$.

Lemma 3. *Let $A^* \in B(m, n)$ be the matrix defined by (14) and $A \in \boldsymbol{A}$ be arbitrary. The following equivalence holds true:*

$$A \otimes \overline{x} \leqslant \overline{b} \quad \textit{if and only if} \quad A \leqslant A^*.$$

Proof. If $A \leqslant A^*$ then

$$[A \otimes \overline{x}]_i \leqslant [A^* \otimes \overline{x}]_i = \max_{j \in N}\{a^*_{ij} + \overline{x}_j\} \leqslant \max_{j \in N}\{\overline{b}_i - \overline{x}_j + \overline{x}_j\} = \overline{b}_i$$

for each $i \in M$.

Conversely, for each $i \in M,\ j \in N$ such that $a^*_{ij} = \overline{a}_{ij}$ the inequality $a_{ij} \leqslant a^*_{ij}$ is trivially satisfied. If $i \in M,\ j \in N$ are such that $a^*_{ij} = \overline{b}_i - \overline{x}_j$ then the inequality $A \otimes \overline{x} \leqslant \overline{b}$ implies $a_{ij} + \overline{x}_j \leqslant \overline{b}_i$ or equivalently $a_{ij} \leqslant \overline{b}_i - \overline{x}_j = a^*_{ij}$. Thus $A \leqslant A^*$. □

In fact, Lemma 3 says that the matrix A^* is the maximum matrix fulfilling inequality (13).

Theorem 7. *Interval system (3) is T1 solvable if and only if interval system (3) is T2 solvable and*

$$A^* \otimes \underline{x} \geqslant \underline{b}. \tag{15}$$

Proof. First we prove that from the T2 solvability of (3) it follows that $A^* \in \boldsymbol{A}$. Inequality (11) implies $\underline{a}_{ij} + \overline{x}_j \leqslant \overline{b}_i$, or equivalently $\overline{b}_i - \overline{x}_j \geqslant \underline{a}_{ij}$, for each $i \in M,\ j \in N$. As $\overline{a}_{ij} \geqslant \underline{a}_{ij}$ for each $i \in M,\ j \in N$, we have $A^* \geqslant \underline{A}$. The inequality $A^* \leqslant \overline{A}$ follows directly from (14).

Lemma 3 and inequality (15) imply that the matrix $A^* \in \boldsymbol{A}$ satisfies the system of inequalities (12), (13). According to Lemma 2 interval system (3) is T1 solvable.

For the converse implication suppose that there exists a matrix $A \in \boldsymbol{A}$ such that A satisfies the system of inequalities (12), (13). By Lemma 3 we get $A \leqslant A^*$ and consequently inequality (12) implies inequality (15).

The implication $T1 \Rightarrow T2$ trivially holds. □

Example 1. Let us have

$$\boldsymbol{A}=\begin{pmatrix}\langle 3,7\rangle & \langle 7,10\rangle & \langle 10,12\rangle\\ \langle 2,9\rangle & \langle 6,12\rangle & \langle 8,15\rangle\\ \langle 5,8\rangle & \langle 2,7\rangle & \langle 7,11\rangle\end{pmatrix},\quad \boldsymbol{x}=\begin{pmatrix}\langle 5,10\rangle\\ \langle 1,3\rangle\\ \langle 1,8\rangle\end{pmatrix},\quad \boldsymbol{b}=\begin{pmatrix}\langle 10,18\rangle\\ \langle 12,16\rangle\\ \langle 13,16\rangle\end{pmatrix}.$$

First, we check the T2 solvability.

Since $\underline{A}\otimes\overline{x}=(18,16,15)^T\leqslant\overline{b}$ and $\overline{A}\otimes\underline{x}=(13,16,13)^T\geqslant\underline{b}$, the given interval system is T2 solvable.

We shall construct the matrix A^*. By (14), we get

$$A^*=\begin{pmatrix}7 & 10 & 10\\ 6 & 12 & 8\\ 6 & 7 & 8\end{pmatrix}.$$

Since $A^*\otimes\underline{x}=(12,13,11)^T\not\geqslant\underline{b}$, the given interval system is not T1 solvable.

5. Weak T6 and strong T6 solvability

In this section we define the notion of a T6-vector and bring necessary and sufficient conditions for the weak T6 and strong T6 solvability.

Definition 2. A vector $b\in\boldsymbol{b}$ is a T6-vector of interval system (3) if for each $x\in\boldsymbol{x}$ there exists $A\in\boldsymbol{A}$ such that $A\otimes x=b$.

Theorem 8. *A vector $b\in\boldsymbol{b}$ is a T6-vector of interval system (3) if and only if*

$$\underline{A}\otimes\overline{x}\leqslant b\leqslant\overline{A}\otimes\underline{x}. \tag{16}$$

Proof. Suppose that a vector $b\in\boldsymbol{b}$ satisfies system of inequalities (16). Let $x\in\boldsymbol{x}$ be arbitrary, but fixed. Denote by A^x the matrix with elements

$$a^x_{ij}=\min\{\overline{a}_{ij},b_i-x_j\} \tag{17}$$

for each $i\in M,\ j\in N$. We prove that $A^x\in\boldsymbol{A}$ and $A^x\otimes x=b$. The inequality $\underline{A}\otimes\overline{x}\leqslant b$ implies $\underline{a}_{ij}+\overline{x}_j\leqslant b_i$ for each $i\in M,\ j\in N$. Then $b_i-x_j\geqslant b_i-\overline{x}_j\geqslant\underline{a}_{ij}$, i.e., $a^x_{ij}\geqslant\underline{a}_{ij}$ for each $i\in M,\ j\in N$. Moreover, the inequality $a^x_{ij}\leqslant\overline{a}_{ij}$ holds, hence we get $A^x\in\boldsymbol{A}$.

For a fixed $i\in M$ denote by N_i the set of indices $\{j\in N:\ \overline{a}_{ij}\geqslant b_i-x_j\}$. The inequality $\overline{A}\otimes\underline{x}\geqslant b$ implies that there exists $j\in N$ such that $\overline{a}_{ij}+\underline{x}_j\geqslant b_i$. Then $b_i-x_j\leqslant b_i-\underline{x}_j\leqslant\overline{a}_{ij}$, so $N_i\neq\emptyset$. For each $j\in N_i$ we have $a^x_{ij}+x_j=b_i$, so $\bigoplus_{j\in N_i}(a^x_{ij}\otimes x_j)=b_i$. If $j\notin N_i$ then $a^x_{ij}+x_j=\overline{a}_{ij}+x_j<b_i-x_j+x_j=b_i$ and consequently $\bigoplus_{j\notin N_i}(a^x_{ij}\otimes x_j)<b_i$. We get $[A^x\otimes x]_i=\bigoplus_{j\in N_i}(a^x_{ij}\otimes x_j)\oplus\bigoplus_{j\notin N_i}(a^x_{ij}\otimes x_j)=b_i$ for each $i\in M$, thus the vector b is a T6-vector of (3).

For the converse implication suppose that $b\in\boldsymbol{b}$ is a T6-vector of (3). Then for $x=\overline{x}$ there exists a matrix $A\in\boldsymbol{A}$ such that $A\otimes\overline{x}=b$. Consequently, $\underline{A}\otimes\overline{x}\leqslant A\otimes\overline{x}=b$, so the first inequality in (16) is satisfied.

Similarly, the second inequality in (16) follows from the fact that for $x=\underline{x}$ there exists $C\in\boldsymbol{A}$ such that $C\otimes\underline{x}=b$ and from the monotonicity of $\otimes$. $\square$

The following theorem provides a necessary and sufficient condition for the weak T6 solvability. Realize that the weak T6 solvability of interval system (3) means that there exists a vector $b \in \boldsymbol{b}$ such that b is a T6-vector of interval system (3).

Theorem 9. *Interval system (3) is weakly T6 solvable if and only if interval system (3) is T2 solvable and*

$$\underline{A} \otimes \overline{x} \leqslant \overline{A} \otimes \underline{x}. \tag{18}$$

Proof. First, we prove the necessary condition. The existence of a T6-vector $b \in \boldsymbol{b}$ implies (16) which implies (18). According to the definitions of the T2 and weak T6 solvability the weak T6 solvability implies the T2 solvability.

For the converse implication suppose that interval system (3) is T2 solvable and inequality (18) is satisfied, but interval system (3) is not weakly T6 solvable, i.e., there is no vector $b \in \boldsymbol{b}$ fulfilling inequality (16). This means that there exists $i \in M$ such that $\langle [\underline{A} \otimes \overline{x}]_i, [\overline{A} \otimes \underline{x}]_i \rangle \cap \langle \underline{b}_i, \overline{b}_i \rangle = \emptyset$. We have two possibilities:

$$\overline{b}_i < [\underline{A} \otimes \overline{x}]_i \quad \text{or} \quad [\overline{A} \otimes \underline{x}]_i < \underline{b}_i.$$

In the first case we get the contradiction with inequality (11), the second case results in contradiction with inequality (10). □

Denote by $b^*,\ b^{**}$ the vectors with entries

$$b_i^* = \max\{[\underline{A} \otimes \overline{x}]_i, \underline{b}_i\}, \tag{19}$$

$$b_i^{**} = \min\{[\overline{A} \otimes \underline{x}]_i, \overline{b}_i\} \tag{20}$$

for each $i \in M$.

Theorem 10. *Let interval system (3) be weakly T6 solvable. A vector $b \in \boldsymbol{b}$ is a T6- vector of (3) if and only if $b \in \langle b^*, b^{**} \rangle$.*

Proof. The proof follows from Theorem 8 and from the second part of the proof of Theorem 9, as $\langle [\underline{A} \otimes \overline{x}]_i, [\overline{A} \otimes \underline{x}]_i \rangle \cap \langle \underline{b}_i, \overline{b}_i \rangle = \langle b_i^*, b_i^{**} \rangle$ for each $i \in M$. □

The following theorem brings the necessary and sufficient conditions for the strong T6 solvability which means that each vector $b \in \boldsymbol{b}$ is a T6-vector of interval system (3).

Theorem 11. *Interval system (3) is strongly T6 solvable if and only if*

$$\underline{A} \otimes \overline{x} \leqslant \underline{b}, \tag{21}$$

$$\overline{A} \otimes \underline{x} \geqslant \overline{b}. \tag{22}$$

Proof. Interval system (3) is strongly T6 solvable if and only if system of inequalities (16) is satisfied for each $b \in \boldsymbol{b}$. The validity of the inequality $\underline{A} \otimes \overline{x} \leqslant b$ ($\overline{A} \otimes \underline{x} \geqslant b$) for each $b \in \boldsymbol{b}$ is equivalent to (21) ((22)). □

6. Weak T7 and strong T7 solvability

In this section we introduce the notion of a T7-vector and prove equivalent conditions for the weak T7 and strong T7 solvability.

Definition 3. A vector $b \in \boldsymbol{b}$ is a T7-vector of interval system (3) if there exists $A \in \boldsymbol{A}$ such that for each $x \in \boldsymbol{x}$ the equality $A \otimes x = b$ holds.

Lemma 4. *A vector $b \in \boldsymbol{b}$ is a T7-vector of interval system (3) if and only if there exists $A \in \boldsymbol{A}$ such that*

$$A \otimes \underline{x} = A \otimes \overline{x} = b. \tag{23}$$

Proof. If $A \in \boldsymbol{A}$ satisfies equality (23), then for each $x \in \boldsymbol{x}$ we have $b = A \otimes \underline{x} \leqslant A \otimes x \leqslant A \otimes \overline{x} = b$ which implies $A \otimes x = b$ for each $x \in \boldsymbol{x}$. So the vector b is T7-vector of (3).

The converse implication is trivial. □

Theorem 12. *A vector b is a T7-vector of interval system (3) if and only if b is a T6-vector of interval system (3) and for each $i \in M$ there exists $j \in N$ such that*

$$\overline{a}_{ij} \otimes \underline{x}_j \geqslant b_i \quad \text{and} \quad \underline{x}_j = \overline{x}_j. \tag{24}$$

Proof. If a vector $b \in \boldsymbol{b}$ is a T7-vector of interval system (3) then equalities (23) are satisfied for some matrix $A \in \boldsymbol{A}$ and consequently for each $i \in M$ there exists $j \in N$ such that

$$a_{ij} \otimes \underline{x}_j = a_{ij} \otimes \overline{x}_j = b_i. \tag{25}$$

Equalities (25) imply (24) immediately.

From Definition 2 and Definition 3 it follows that if a vector $b \in \boldsymbol{b}$ is a T7-vector of (3) then the vector b is a T6-vector of (3).

For the converse implication suppose that a vector $b \in \boldsymbol{b}$ is a T6-vector of (3) and (24) is fulfilled. We shall construct the matrix $\tilde{A} \in \boldsymbol{A}$ such that $\tilde{A}$ fulfills system of equalities (23).

Let $i \in M$ be fixed. Denote $N_i = \{j \in N : \ \overline{a}_{ij} \otimes \underline{x}_j \geqslant b_i \text{ and } \underline{x}_j = \overline{x}_j\}$. Define the matrix $\tilde{A} = (\tilde{a}_{ij})$ as follows:

$$\tilde{a}_{ij} = \begin{cases} b_i - \underline{x}_j & \text{for } j \in N_i, \\ \underline{a}_{ij} & \text{for } j \notin N_i. \end{cases} \tag{26}$$

For $j \in N_i$ we have $b_i - \underline{x}_j \leqslant \overline{a}_{ij}$ and $b_i - \underline{x}_j = b_i - \overline{x}_j \geqslant \underline{a}_{ij}$ (from (16)). Thus $\tilde{A} \in \boldsymbol{A}$.

For $j \in N_i$ we have $\tilde{a}_{ij} + \underline{x}_j = \tilde{a}_{ij} + \overline{x}_j = b_i$.

For $j \notin N_i$ the inequalities $\tilde{a}_{ij} \otimes \underline{x}_j \leqslant \tilde{a}_{ij} \otimes \overline{x}_j = \underline{a}_{ij} \otimes \overline{x}_j \leqslant b_i$ follow from (16). Therefore $\tilde{A} \otimes \underline{x} = \tilde{A} \otimes \overline{x} = b$ and by Lemma 4 the vector b is a T7-vector of (3). □

Corollary 1. *If $\underline{x}_j < \overline{x}_j$ for each $j \in N$ then interval system (3) is not weakly T7 solvable.*

Proof. The proof follows immediately from Theorem 12. □

Theorem 13. *Interval system (3) is weakly T7 solvable if and only if interval system (3) is weakly T6 solvable and the vector b^* defined by (19) is a T7-vector of interval system (3).*

Proof. First we prove the necessary condition.

If interval system (3) is not weakly T6 solvable then interval system (3) is not weakly T7 solvable.

If the vector b^* is not a T7-vector then by Theorem 12 either b^* is not a T6-vector or there exists $i \in M$ such that for each $j \in N$ we have $\overline{a}_{ij} \otimes \underline{x}_j < b_i^*$ or $\underline{x}_j \neq \overline{x}_j$. In the first case by Theorem 10 interval system (3) is not weakly T6 solvable and consequently interval system (3) is not weakly T7 solvable. In the second case for each $b \in \langle b^*, b^{**} \rangle$ and for each $j \in N$ we get $\overline{a}_{ij} \otimes \underline{x}_j < b_i$ or $\underline{x}_j \neq \overline{x}_j$, so there is no T7-vector of interval system (3). Thus interval system (3) is not weakly T7 solvable.

The converse implication is trivial. □

Theorem 14. *Interval system (3) is strongly T7 solvable if and only if interval system (3) is strongly T6 solvable and the vector $\underline{b}$ is a T7-vector of interval system (3).*

Proof. Suppose that interval system (3) is strongly T6 solvable, i.e., $\langle b^*, b^{**}\rangle = \boldsymbol{b}$ and $\overline{b}$ is a T7-vector of interval system (3). By Theorem 12 for each $i \in M$ there exists $j \in N$ such that $\overline{a}_{ij} \otimes \underline{x}_j \geqslant \overline{b}_i \geqslant b_i$ and $\underline{x}_j = \overline{x}_j$. It follows that each vector $b \in \boldsymbol{b}$ is a T7-vector of (3). Thus interval system (3) is strongly T7 solvable.

The converse implication is trivial. □

Example 2. Let us have

$$\boldsymbol{A} = \begin{pmatrix} \langle 1, 9\rangle & \langle 3, 6\rangle & \langle 2, 5\rangle \\ \langle 4, 7\rangle & \langle 6, 9\rangle & \langle 2, 6\rangle \\ \langle 2, 5\rangle & \langle 3, 8\rangle & \langle 1, 5\rangle \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \langle 3, 6\rangle \\ \langle 5, 5\rangle \\ \langle 4, 7\rangle \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \langle 9, 12\rangle \\ \langle 12, 14\rangle \\ \langle 10, 12\rangle \end{pmatrix}.$$

We check the solvability concepts defined in the present paper.
T2 solvability: $\underline{A} \otimes \overline{x} = (9, 11, 8)^T \leqslant \overline{b}$ and $\overline{A} \otimes \underline{x} = (12, 14, 13)^T \geqslant \underline{b}$.
Answer: By Theorem 6 the given interval system is T2 solvable.

Weak T6 solvability: $\underline{A} \otimes \overline{x} \leqslant \overline{A} \otimes \underline{x}$.
Answer: By Theorem 9 the given interval system is weakly T6 solvable with $b^* = \underline{b},\ b^{**} = \overline{b}$.

Strong T6 solvability: $\underline{A} \otimes \overline{x} \leqslant \underline{b}$ and $\overline{A} \otimes \underline{x} \geqslant \overline{b}$.
Answer: By Theorem 11 the given interval system is strongly T6 solvable.

Weak T7 solvability:
For $i = 1,\ j = 2$ we have $\overline{a}_{12} \otimes \underline{x}_2 \geqslant b_1^*$ and $\underline{x}_2 = \overline{x}_2$,
for $i = 2,\ j = 2$ we have $\overline{a}_{22} \otimes \underline{x}_2 \geqslant b_2^*$ and $\underline{x}_2 = \overline{x}_2$,
for $i = 3,\ j = 2$ we have $\overline{a}_{32} \otimes \underline{x}_2 \geqslant b_3^*$ and $\underline{x}_2 = \overline{x}_2$.
Answer: By Theorem 12 vector b^* is a T7-vector and by Theorem 13 the given interval system is weakly T7 solvable.

Strong T7 solvability:
For $i = 1$ there does not exist $j \in N$ such that $\overline{a}_{1j} \otimes \underline{x}_j \geqslant \overline{b}_1$ and $\underline{x}_j = \overline{x}_j$.
Answer: By Theorem 14 the given interval system is not strongly T7 solvable.

References

[1] K. Cechlárová, R.A. Cuninghame-Green, Interval systems of max-separable linear equations, Linear Algebra Appl. 340 (2002) 215–224.
[2] R.A. Cuninghame-Green, Minimax algebra, Springer, Berlin, 1979.
[3] M. Gavalec, J. Plavka, Monotone interval eigenproblem in max–min algebra, Kybernetika 46 (3) (2010) 387–396.
[4] H. Myšková, Interval systems of max-separable linear equations, Linear Algebra Appl. 403 (2005) 263–272.
[5] H. Myšková, Control solvability of interval systems of max-separable linear equations, Linear Algebra Appl. 416 (2006) 215–223.
[6] H. Myšková, The algorithm for testing solvability of max-plus interval systems, in: Proceeding of 28th International Conference on Mathematical Methods in Economics 2011, Jánska dolina, Slovakia, September 06–09, 2011, pp. 498–503.
[7] H. Myšková, On the algorithm for testing T4 solvability of max-plus interval systems, Kybernetika, in press.
[8] W. Oettli, W. Prager, Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides, Numer. Math. 6 (1964) 405–409.
[9] G.J. Olsder et al., Course notes: max-algebra approach to discrete event systems, in: Algebras Max-Plus et Applications an Informatique et Automatique, INRIA, 1998, pp. 147–196.
[10] J. Plavka, On the $O(n^3)$ algorithm for checking the strong robustness of interval fuzzy matrices, Discrete Appl. Math. 160 (2012) 640–647.
[11] J. Rohn, Systems of interval linear equations and inequalities (rectangular case), Institute of Computer Science, Academy of Sciences of the Czech Republic, Technical Report No. 875, September 17, 2002.
[12] N.N. Vorobjov, Extremal algebra of positive matrices, Datenverarb. Kybernetik 3 (1967) 39–71 (in Russian).